A Note on Reaction-Diffusion Systems with Skew-Gradient Structure

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Abstract

Reaction-diffusion systems with skew-gradient structure can be viewed as a sort of activator-inhibitor systems. We use variational methods to study the existence of steady state solutions. Furthermore, there is a close relation between the stability of a steady state and its relative Morse index. Some numerical results will also be disussed.

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1 Introduction

In this note we consider reaction-diffusion systems of the form

$$M_1 u_t = D_1 \Delta u + F_1(u,v), \qquad (1.1)$$

$$M_2 v_t = D_2 \Delta v - F_2(u,v),$$

$$x \in \Omega,\ t > 0. \qquad (1.2)$$

Here Ω is a smooth bounded domain in $\mathbb{R}^n$, $u(x,t)$ is an m_1-dimensional vector function, $v(x,t)$ is an m_2-dimensional vector function, M_1, M_2, D_1 and D_2 are positive definite matrices, and there exists a function F such that $\nabla F = (F_1, F_2)$. Such systems can be viewed as a sort of activator-inhibitor systems.

A well-known example is

$$u_t = d_1 \Delta u + f(u) - v, \qquad (1.3)$$

$$\tau v_t = d_2 \Delta v + \sigma u - \gamma v, \qquad (1.4)$$

where $d_1, d_2, \sigma, \gamma, \tau \in (0, \infty)$ and f is a cubic polynomial. The case of $d_2 = 0$ has been considered as a model for the Hodgkin-Huxley system [13, 22] to describe the behavior of electrical impulses in the axon of the squid. More recently, several variations of this system appeared in neural net models for short-term memory and in nerve cells of heart muscle.

As in [29], (1.1)-(1.2) will be referred as a skew-gradient system in which a steady state is a critical point of

$$\Phi(u,v) = \int_\Omega \frac{1}{2}(D_1\nabla u,\nabla u) - \frac{1}{2}(D_2\nabla v,\nabla v) - F(u,v)dx. \qquad (1.5)$$

A steady state $(\bar{u},\bar{v})$ is called a mini-maximizer of Φ if $\bar{u}$ is a local minimizer of $\Phi(\cdot,\bar{v})$ and $\bar{v}$ is a local maximizer of $\Phi(\bar{u},\cdot)$. It has been shown [29] that non-degenerate mini-maximizers of Φ are linearly stable. This result gives a natural generalization of a stability criterion for the gradient system in which all the non-degenerate local minimizers are stable steady states.

A remarkable property proved in [29] is that any mini-maximizer must be spatially homogeneous if Ω is a convex set. This kind of results have been established by Casten and Holland [5] and Matano [20] for the scalar reaction-diffusion equation, and generalized by Jimbo and Morita [15] and Lopes [19] for the gradient system. In case Ω is symmetric with respect to x_j, Lopes [19] showed that a global minimizer of gradient system is symmetric with respect to x_j; while Chen [7] obtained parallel results for the global mini-maximizers in the skew-gradient system.

In connection with calculus of variations, there is a close relation between the stability of a steady state of skew-gradient system and its relative Morse index. Based on this idea, some stability criteria for the steady states of (1.1)-(1.2) are illustrated in section 2. In section 3, variational arguments are used to study the existence of steady states and their relative Morse indices. Section 4 contains numerical investigation of skew-gradient systems. A particular example to be studied is

$$\begin{aligned} u_t &= d_1u_{xx} + f(u) - v - w, \\ \tau_2 v_t &= d_2 v_{xx} + u - \gamma_2 v, \\ \tau_3 w_t &= d_3 w_{xx} + u - \gamma_3 w, \end{aligned}$$

which served as a model [4] for gas-discharge systems.

2 Stability Criteria

Let E be a Hilbert space. For a closed subspace U of E, P_U denotes the orthogonal projection from E to U and $U^\perp$ denotes the orthogonal complement of U. For two closed subspaces U and W of E, denoted by $U \sim W$ if $P_U - P_W$ is a compact operator. In this case, both $W \cap U^\perp$ and $W^\perp \cap U$ are of finite dimensional. The relative dimension of W with respect to U is defined by

$$dim(W,U) = dim(W \cap U^\perp) - dim(W^\perp \cap U). \quad (2.1)$$

If A is a self-adjoint Fredholm operator on E, there is a unique A-invariant orthogonal splitting

$$E = E_+(A) \oplus E_-(A) \oplus E_0(A)$$

with $E_+(A)$, $E_-(A)$ and $E_0(A)$ being respectively the subspaces on which A is positive definite, negative definite and null. For a pair of self-adjoint Fredholm operators A and $\bar{A}$, it will be denoted by $A \sim \bar{A}$ if $E_-(A) \sim E_-(\bar{A})$. In this case, a relative Morse index $i(A, \bar{A})$ is defined by

$$i(A, \bar{A}) = dim(E_-(\bar{A}), E_-(A)). \quad (2.2)$$

We refer to [1] for more details of relative Morse index.

For a critical point $(\bar{u}, \bar{v})$ of Φ, let $\Phi''(\bar{u}, \bar{v})$ denote the second Frechet derivative of Φ at $(\bar{u}, \bar{v})$. A critical point $(\bar{u}, \bar{v})$ is called non-degenerate if the null space of $\Phi''(\bar{u}, \bar{v})$ is trivial. Let

$$M = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}, D = \begin{pmatrix} -D_1 & 0 \\ 0 & D_2 \end{pmatrix},$$

$$Q = \begin{pmatrix} -I_{m_1} & 0 \\ 0 & I_{m_2} \end{pmatrix}$$

and I_k be the $k \times k$ identity matrix. For a gradient system, it is known that a non-degenerate critical point with non-zero Morse index is an unstable steady state. The next theorem gives a parallel result for the skew-gradient system.

Theorem 1. Suppose $i(-Q, \Phi''(\bar{u}, \bar{v})) \neq 0$ and $dimE_0(\Phi''(\bar{u}, \bar{v})) = 0$, then for any positive definite matrices M_1 and M_2, $(\bar{u}, \bar{v})$ is an unstable steady state of (1.1)-(1.2).

In [29] Yanagida pointed out an interesting property that a non- degenerate minimaximizer of Φ is always stable for any positive matrices M_1 and M_2 given in (1.1)-(1.2). An interesting question is whether there exist steady states with stability depending on the reaction rates of the system. Let P^+ and P^- be the orthogonal projections from E to $E_+(Q)$ and $E_-(Q)$ respectively. Define $\Psi_0 = M^{-\frac{1}{2}}(D\Delta - \nabla^2 F(\bar{u}, \bar{v}))M^{-\frac{1}{2}}$, $\psi_1 = P^- \Psi_0 P^-$

and $\psi_2 = P^+\Psi_0 P^+$. Set $m = m_1 + m_2$, $\mathfrak{D} = H^2(\Omega, \mathbb{R}^m)$,

$$\rho_i(\psi_1) = \inf_{z\in\mathfrak{D}} \frac{\langle \psi_1 z, z\rangle_{L^2}}{\|P^- z\|_{L^2}^2} \tag{2.3}$$

and

$$\rho_s(\psi_2) = \sup_{z\in\mathfrak{D}} \frac{\langle \psi_2 z, z\rangle_{L^2}}{\|P^+ z\|_{L^2}^2}. \tag{2.4}$$

Theorem 2. Assume that $i(-Q, \Phi''(\bar{u}, \bar{v})) = 0$ and $dim E_0(\Phi''(\bar{u}, \bar{v})) = 0$. Then $(\bar{u}, \bar{v})$ is stable if $\rho_i(\psi_1) > \rho_s(\psi_2)$.

Remark. In case we treat the Dirichlet boundary condition $u|_{\partial\Omega} = v|_{\partial\Omega} = 0$, $\mathfrak{D}$ is replaced by $H^2(\Omega, \mathbb{R}^m) \bigcap H_0^1(\Omega, \mathbb{R}^m)$.

The proofs of Theorem 1 and Theorem 2 can be found in [8].

3 Applications of Theorem 1 and Theorem 2

In dealing with a strongly indefinite functional Φ, a critical point theorem established by Benci and Rabinowitz [3] can be used to obtain steady states of (1.1)-(1.2).

Theorem 3. Let E be a separable Hilbert space with an orthogonal splitting $E = W_+ \oplus W_-$, and $B_r = \{\xi | \xi \in E, \|\xi\| < r\}$. Assume that $\Phi(\xi) = \frac{1}{2}\langle \hat{\Lambda}\xi, \xi\rangle + b(\xi)$, where $\hat{\Lambda}$ is a self-adjoint invertible operator on E, $b \in C^2(E, \mathbb{R})$ and b' is compact. Set $S = \partial B_\rho \cap W_+$ and $N = \{\xi^- + se | \xi^- \in B_r \cap W_- \text{ and } s \in [0, \bar{R}]\}$, where $e \in \partial B_1 \cap W_+$, $r > 0$ and $\bar{R} > \rho > 0$. If Φ satisfies (PS)* condition and $\sup_{\partial N} \Phi < \inf_S \Phi$, then Φ possesses a critical point $\bar{\xi}$ such that $\inf_S \Phi \geq \Phi(\bar{\xi}) \geq \sup_{\partial N} \Phi$. Moreover, if $W_- \sim E_-$, then

$$\begin{aligned} i(\hat{\Lambda}, \Phi''(\bar{\xi})) &\leq dim(W_-, E_-) + 1 \leq i(\hat{\Lambda}, \Phi''(\bar{\xi})) \\ &+ dim E_0(\Phi''(\bar{\xi})). \end{aligned} \tag{3.1}$$

Remark. (a) See e.g. [2, 8] for the definition of $(PS)^*$ condition.
(b) The index estimates (3.1) were obtained by Abbondandolo and Molina [2].

In a demonstration of using Theorem 3 to study the existence and stability of steady state solutions, we consider a perturbed FitzHugh-Nagumo system in the first example :

$$u_t = d_1 \Delta u + f(u) - v, \tag{3.2}$$

$$\tau v_t = d_2 \Delta v + u - \gamma v - h(v). \tag{3.3}$$

A steady state of (3.2)-(3.3) is a critical point of

$$\Phi(u,v) = \int_\Omega \frac{d_1}{2}|\nabla u|^2 - \frac{d_2}{2}|\nabla v|^2 - F(u,v)dx,$$

where

$$\begin{aligned} F(u,v) &= -(\frac{1}{4}u^4 - \frac{\beta+1}{3}u^3 + \frac{\beta}{2}u^2) - uv \\ &\quad + \frac{\gamma}{2}v^2 + H(v), \end{aligned} \tag{3.4}$$

$\beta \in (0, \frac{1}{2})$ and $H(v) = \int_0^v h(y)dy$. It is assumed that $\gamma > 9(2\beta^2 - 5\beta + 2)^{-1}$, and h satisfies the following condition:

(h1) $h \in C^1$, $h(0) = h'(0) = 0$ and $yh(y) \geq 0$ for $y \in \mathbb{R}$.

Define

$$\Lambda = \begin{pmatrix} -d_1\Delta - f'(0) & 1 \\ 1 & d_2\Delta - \gamma \end{pmatrix}.$$

Let

$$\begin{aligned} \mu_k^+ &= \frac{1}{2}\big[(d_1-d_2)\lambda_k-(f'(0)+\gamma) \\ &\quad +\sqrt{((d_1+d_2)\lambda_k-f'(0)+\gamma)^2+4}\big] \end{aligned}$$

and

$$\begin{aligned} \mu_k^- &= \frac{1}{2}\big[(d_1-d_2)\lambda_k-(f'(0)+\gamma) \\ &\quad -\sqrt{((d_1+d_2)\lambda_k-f'(0)+\gamma)^2+4}\big], \end{aligned}$$

where $\{-\lambda_k\}$ are the eigenvalues of the Laplace operator and $\{\phi_k\}$ are the corresponding eigenfunctions. By straightforward calculation $\Lambda e_k^+\phi_k = \mu_k^+ e_k^+\phi_k$ and $\Lambda e_k^-\phi_k = \mu_k^- e_k^-\phi_k$, where

$$\begin{aligned} e_k^+ &= (1, \frac{1}{2}[\sqrt{((d_1+d_2)\lambda_k - f'(0)+\gamma)^2+4} \\ &\quad - [(d_1+d_2)\lambda_k - f'(0)+\gamma]]), \end{aligned}$$

$$\begin{aligned} e_k^- &= (1, \frac{-1}{2}[(d_1+d_2)\lambda_k - f'(0)+\gamma \\ &\quad + \sqrt{((d_1+d_2)\lambda_k - f'(0)+\gamma)^2+4}]). \end{aligned}$$

It is clear that $\mu_k^+ > 0$ and $\mu_k^- < 0$ for all $k \in \mathbb{N}$. Let $E_+ = \oplus_{k=1}^\infty V_k^+$ and $E_- = \oplus_{k=1}^\infty V_k^-$, where $V_k^+ = \{s\phi_k e_k^+ | s \in \mathbb{R}\}$ and $V_k^- = \{s\phi_k e_k^- | s \in \mathbb{R}\}$. Define $\Lambda^+ = \Lambda|_{E_+}$, $\Lambda^- = \Lambda|_{E_-}$ and

$$\begin{aligned} \langle \hat{\Lambda} z_1, z_2\rangle &= \int_\Omega ((\Lambda^+)^{\frac{1}{2}}z_1, (\Lambda^+)^{\frac{1}{2}}z_2) \\ &\quad - ((\Lambda^-)^{\frac{1}{2}}z_1, (\Lambda^-)^{\frac{1}{2}}z_2)dx \end{aligned}$$

for $z_1, z_2 \in E$. As an application of Theorem 3, we have the following result.

Theorem 4. Let $\mathcal{B}_R$ be a ball in $\mathbb{R}^n$ with radius R. If Ω contains a ball $\mathcal{B}_R$ with R being sufficiently large, then there exists a steady state $(\bar{u}, \bar{v})$ of (3.2)-(3.3), and

$$\begin{aligned} & i(-Q, \Phi''(\bar{u},\bar{v}) \leq 1 \leq i(-Q, \Phi''(\bar{u},\bar{v})) \\ & + dim E_0(\Phi''(\bar{u},\bar{v})). \end{aligned}$$

In view of Theorem 1, $(\bar{u}, \bar{v})$ is unstable if it is a non-degenerate critical point of Φ. More details can be found in [8].

We now turn to some examples to seek stable steady states of skew-gradient systems. Consider

$$u_t = \triangle u - u - v, \tag{3.5}$$

$$\tau v_t = \triangle v + 2v + u - |v|v. \tag{3.6}$$

Straightforward calculation gives

$$\Lambda = \begin{pmatrix} -\triangle+1 & 1 \\ 1 & \triangle+2 \end{pmatrix},$$

$\mu_k^+ = \frac{1}{2}(3+\sqrt{(2\lambda_k-1)^2+4})$ and $\mu_k^- = \frac{1}{2}(3-\sqrt{(2\lambda_k-1)^2+4})$. It is clear that $\mu_k^+ > 0$ for all $k \in \mathbb{N}$. Suppose Ω is a bounded domain in which the eigenvalue distribution of the Laplace operator (under homogeneous Dirichlet boundary conditions) satisfies the following property:

$$\lambda_1 < \frac{1}{2}(\sqrt{5}+1) < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_k \cdots$$

Then it is easily seen that $\mu_1^- > 0$, and $\mu_k^- < 0$ if $k \geq 2$. It follows that $i(-Q,\Lambda) = -1$.

Theorem 5. There is a non-constant steady state $(\bar{u},\bar{v})$ of (3.5)-(3.6). Moreover, if $dim(\Phi''(\bar{u},\bar{v})) = 0$ and $\tau \geq \frac{2-\lambda_1}{1+\lambda_1}$, then $(\bar{u},\bar{v})$ is stable.

In the next example, consider (1.3)-(1.4) with $f(u) = \alpha u - u^3$ and $\sigma = 1$. Suppose there is a $j \in \mathbb{N}$ such that if

$$d_1\lambda_j + \frac{1}{d_2\lambda_j+\gamma} < \alpha < \inf\{d_1\lambda_k + \frac{1}{d_2\lambda_k+\gamma} | k \in \mathbb{N}\backslash\{j\}\} \tag{3.7}$$

By direct calculation $\mu_j^+ < 0$ and $\mu_k^+ > 0$ for $k \in \mathbb{N}\backslash\{j\}$. Also, $\mu_k^- < 0$ for all $k \in \mathbb{N}$. Hence $i(-Q,\Lambda) = 1$. Applying Theorem 3 yields a steady state $(\bar{u},\bar{v})$ of (1.3)-(1.4). Furthermore,

$$i(-Q,-\Phi''(\bar{u},\bar{v}) \leq 0 \leq i(-Q,-\Phi''(\bar{u},\bar{v})) + dimE_0(\Phi''(\bar{u},\bar{v})).$$

This implies that $i(-Q,-\Phi''(\bar{u},\bar{v})) = 0$ if $(\bar{u},\bar{v})$ is a non-degenerate critical point of Φ. Then by Theorem 2, $(\bar{u},\bar{v})$ is stable if $\tau < \frac{\gamma}{\alpha}$. In case of dealing with homogeneous Neumann boundary conditions, $(\bar{u},\bar{v})$ is a spatially inhomogeneous steady state if (3.7) holds for $j \geq 2$. In other words, there exists a stable pattern for (1.3)-(1.4).

For the FitzHugh-Nagumo system, the steady state solutions satisfy

$$d_1\Delta u + f(u) - v = 0, \tag{3.8}$$

$$\frac{d_2}{\sigma}\Delta v + u - \frac{\gamma}{\sigma}v = 0, \tag{3.9}$$

where $f(u) = (1-u)(u-\beta)u$, $\beta \in (0,\frac{1}{2})$. If $\mathcal{L} = \sigma^{-1}(-d_2\Delta+\gamma)^{-1}$ under homogeneous Dirichlet (respectively Neumann) boundary conditions, then for any critical point $\bar{u}$ of

$$\psi(u) = \int_\Omega [\frac{d_1}{2}(|\nabla u|^2 + u\mathcal{L}u) - \int_0^u f(\zeta)d\zeta]dx,$$

$(\bar{u}, \mathcal{L}\bar{u})$ is a steady state of FitzHugh-Nagumo system. In view of the fact that $\sigma \int_\Omega u\mathcal{L}udx = \int_\Omega d_2|\nabla v|^2 + \gamma v^2 dx$, it is easily seen that ψ is bounded from below. In addition to minimizers, the Mountain Pass Lemma has been used to obtain non-trivial solutions [9, 10, 11, 17, 21, 24, 28, 32] of (3.8)-(3.9)

Let u be a critical point of ψ. Straightforward calculation yields

$$\psi''(u) = -\Delta + \mathcal{L} - f'(u),$$

where ψ'' is the second Frechet derivative of ψ and the Morse index of u will be denoted by $i_*(\psi''(u))$. On the other hand, $(u, \mathcal{L}u)$ is also a critical point of

$$\begin{aligned}\Phi(u,v) &= \int_\Omega [\frac{d_1}{2}|\nabla u|^2 - \frac{d_2}{2\sigma}|\nabla u|^2 + uv \\ &\quad - \frac{\gamma}{2\sigma}v^2 - \int_0^u f(\xi)d\xi]dx.\end{aligned}$$

Proposition 1. If u is a critical point of ψ and $v = \mathcal{L}u$, then

$$dimE_0(\psi''(u)) = dimE_0(\Phi''(u,v))$$

and

$$i_*(\psi''(u)) = i(-Q, \Phi''(u,v)).$$

We refer to [8] for a proof of Proposition 1.

For a critical point u obtained by the Mountain Pass Lemma, it is known [6] that

$$i_*(\psi''(u)) \le 1 \le i_*(\psi''(u)) + dimE_0(\psi''(u)).$$

Then by Proposition 1

$$\begin{aligned}i(-Q, \Phi''(u,\mathcal{L}u)) \le 1 \le i(-Q, \Phi''(u,\mathcal{L}u)) \\ + dimE_0(\Phi''(u,\mathcal{L}u)).\end{aligned}$$

Thus if $dimE_0(\psi''(u)) = 0$, it follows from Theorem 1 that $(u, \mathcal{L}u)$ is an unstable steady state of (1.3)-(1.4).

Let $\hat{\psi}_1 = P^-(D\Delta - \nabla^2 F(\bar{u},\bar{v}))P^-$, $\hat{\psi}_2 = P^+(D\Delta - \nabla^2 F(\bar{u},\bar{v}))P^+$,

$$\rho_i(\hat{\psi}_1) = \inf_{z\in\mathfrak{D}} \frac{\langle \hat{\psi}_1 z, z\rangle_{L^2}}{\|P^- z\|^2_{L^2}} \tag{3.10}$$

and

$$\rho_s(\hat{\psi}_2) = \sup_{z\in\mathfrak{D}} \frac{\langle \hat{\psi}_2 z, z\rangle_{L^2}}{\|P^+ z\|^2_{L^2}}. \tag{3.11}$$

Theorem 6. Assume that $i(-Q, \Phi''(\bar{u},\bar{v})) = 0$ and $dimE_0(\Phi''(\bar{u},\bar{v})) = 0$. Then $(\bar{u},\bar{v})$ is stable if one of the following conditions holds:

(i) $\rho_i(\hat{\psi}_1) > 0$, $\rho_s(\hat{\psi}_2) \ge 0$ and

$$\frac{\rho_s(\hat{\psi}_2)}{\rho_i(\hat{\psi}_1)} < \|M_2^{-1}\|^{-1}\|M_1\|^{-1}.$$

(ii) $\rho_i(\hat{\psi}_1) \le 0$, $\rho_s(\hat{\psi}_2) < 0$ and

$$\frac{\rho_i(\hat{\psi}_1)}{\rho_s(\hat{\psi}_2)} < \|M_1^{-1}\|^{-1}\|M_2\|^{-1}.$$

Theorem 6 directly follows from Theorem 2. We refer to [8] for the detail.

If u is a non-degenerate minimizer of ψ and $v = \mathcal{L}u$, then Proposition 1 implies that $i(-Q, \Phi(u,v)) = 0$. Notice that

$$D\Delta - \nabla^2 F(u,v) = \begin{pmatrix} -d_1\Delta - f'(u) & 1 \\ 1 & \frac{d_2}{\sigma}\Delta - \frac{\gamma}{\sigma} \end{pmatrix}.$$

Since $f'(\xi) = -3\xi^2 + 2(\beta+1)\xi - \beta \leq (\beta^2 - \beta + 1)/3$, it easy to check that $\rho_i(\hat{\psi}_1) = \rho_i(-d_1\Delta - f'(u)) \geq d_1\lambda_1 - \frac{(\beta^2-\beta+1)}{3}$ and $\rho_s(\hat{\psi}_2) = \rho_s(\frac{d_2}{\sigma}\Delta - \frac{\gamma}{\sigma}) \leq -(d_2\lambda_1 + \gamma)/\sigma$, where $\lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_k < \cdots$ are the eigenvalues of $-\Delta$. If $\rho_i(\hat{\psi}_1) \leq 0$ and $\tau < \frac{3(d_2\lambda_1+\gamma)}{\sigma((\beta^2-\beta+1)-3d_1\lambda_1)}$, condition (ii) of Theorem 6 holds and consequently (u,v) is a stable steady state of (1.3)-(1.4).

4 Numerical Results

We report some numerical work on the skew-gradient systems, and compare with the theoretical results.

4.1

We start with the following reaction-diffusion system :

$$\begin{aligned} u_t &= d_1u_{xx} + u(u-\beta)(1-u) \\ &\quad - v - w, \qquad (4.1) \\ \tau_2 v_t &= d_2 v_{xx} + u - \gamma_2 v, \qquad (4.2) \\ \tau_3 w_t &= d_3 w_{xx} + u - \gamma_3 w, \\ &\quad x \in (0,1), t > 0. \qquad (4.3) \end{aligned}$$

where $\beta = 0.3$, $\gamma_2 = 1$, $\gamma_3 = 20$, and the homogeneous Neumann boundary conditions will be under consideration. In (4.1)-(4.3), u can be viewed as an activator while v and w act as inhibitors. In view of the theoretical results mentioned in the previous sections, we look for the pattern formation for (4.1)-(4.3) in case the diffusion rate of the activator is small ($d_1 = 10^{-6}$).

By taking $d_2 = 1$ and $d_3 = 10^{-6}$, various types of spatially inhomogeneous steady states have been observed through numerical calculation. In Figure 1 and Figure 3, there is one peak on the profile of u; the one in Figure 1 is symmetric with respect to the spatial variable, while the other is not. We found also instances of steady states with two peaks on the profile of u; but the distance between peaks can be different. We remark based on numerical observation that, with $\tau_2 = \tau_3 = 10^{-4}$, such inhomogeneous steady states are stable under the flow generated by (4.1)-(4.3). Moreover, the solution profiles tell that w is roughly equal to $\gamma_3^{-1} u$ in magnitude.

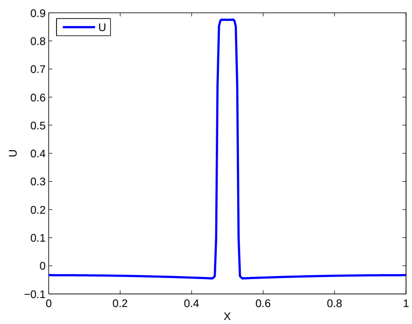

Figure 1: solution profile of u

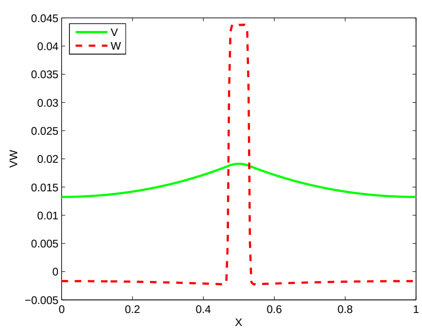

Figure 2: profiles of v and w

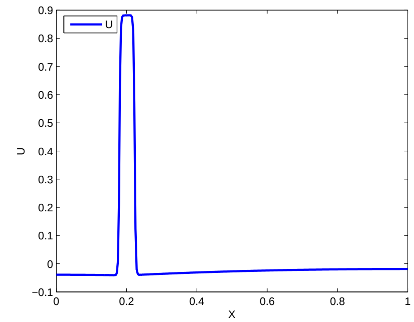

Figure 3: solution profile of u

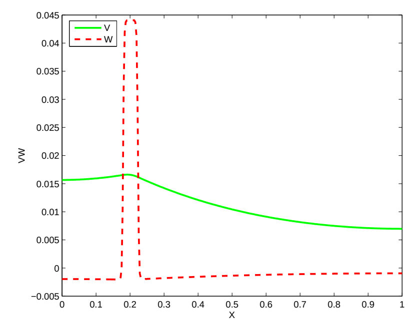

Figure 4: profiles of v and w

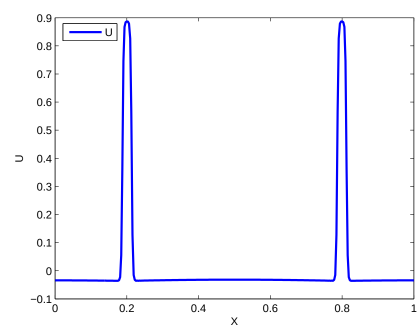

Figure 5: solution profile of u

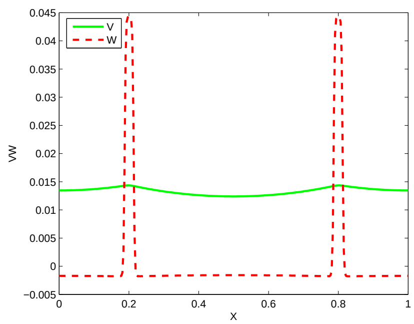

Figure 6: profiles of v and w

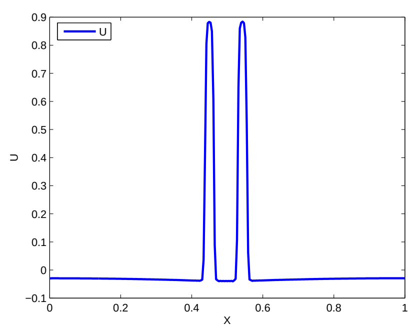

Figure 7: solution profile of u

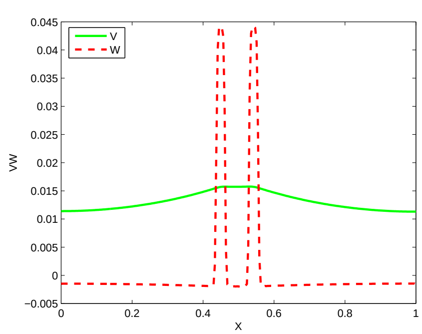

Figure 8: profiles of v and w

We next turn to the case when both inhibitors v and w are acting with large diffusion ($d_2 = d_3 = 1$). As show in Figure 9-10, the pulse (or peak of u) becomes wider. The fact that $\gamma_3 > \gamma_2$ results in $v > w$.

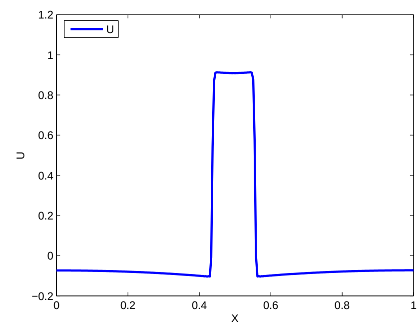

Figure 9: solution profile of u

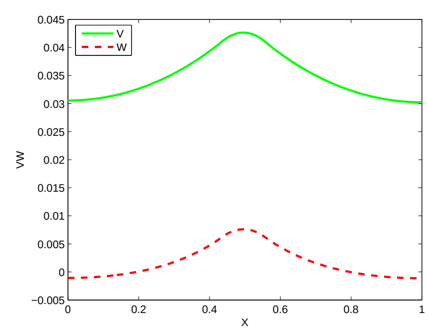

Figure 10: profiles of v and w

Keeping $d_3 = 1$ and reducing d_2 to 10^{-1}, we obtain a stable steady state with rather different profiles as shown in Figure 11-12.

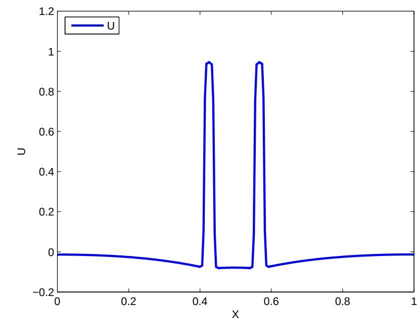

Figure 11: solution profile of u

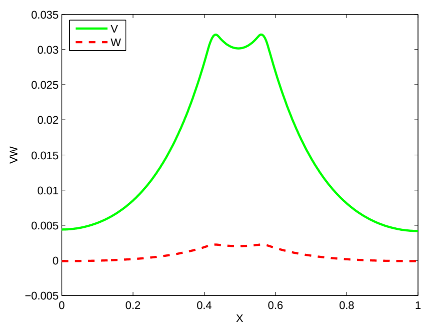

Figure 12: profiles of v and w

4.2

In this subsection we come back to the reaction-diffusion system

$$
\begin{aligned}
u_t &= u_{xx} - u - v, \\
\tau v_t &= v_{xx} + 2v + u - |v|v, \\
& \quad x \in (0,3), t > 0, \\
u(0,t) &= v(0,t) = u(3,t) = v(3,t) = 0.
\end{aligned}
$$

As we know from Theorem 5, the choice of $\tau = 0.1$ leads the flow converging to a non-constant steady state (Figure 13). The behavior in the phase plane of the state variables, at the midpoint of the domain ($x = 1.5$), exhibits a spiral-inward convergence (Figure 14).

On the other hand, we conjecture that such a non-constant steady state become unstable if the value of τ is taking much smaller. Indeed, when $\tau = 0.005$, we observed a time-periodic attractor (Figure 15-16) .

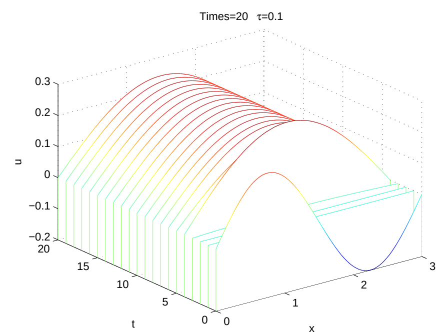

Figure 13: flow of u with $\tau = 0.1$

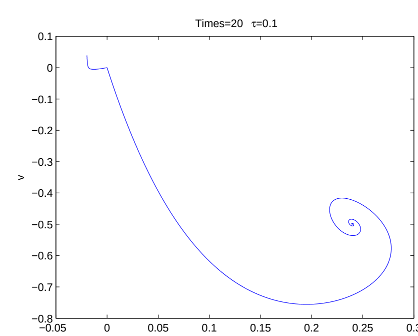

Figure 14: the trajectory of $(u(1.5,t), v(1.5,t))$

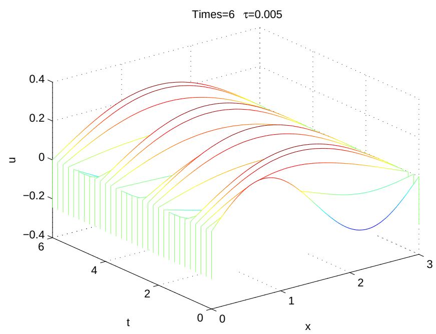

Figure 15: flow of u with $\tau = 0.005$

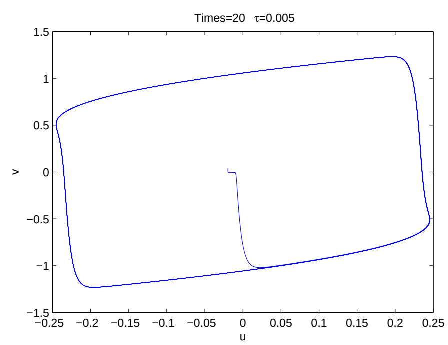

Figure 16: the trajectory of $(u(1.5,t), v(1.5,t)$

The convergence history of the two calculated state variables is recorded in Figure 17-18, which strongly suggests the existence of a stable time-periodic solution. The change of stability seems to result from a Hopf bifurcation and deserves further investigation.

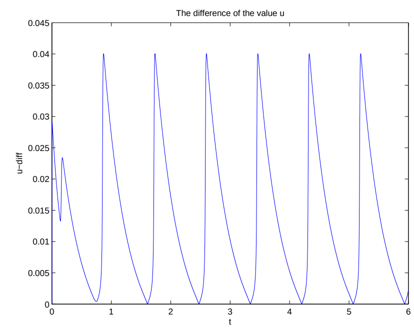

Figure 17: Historic space-accumulated l_1-difference of u

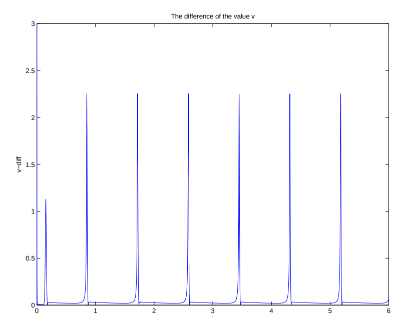

Figure 18: Historic space-accumulated l_1-difference of v

Acknowledgments

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